

Version 1.0



**General Certificate of Education (A-level)**  
**June 2013**

**Mathematics**

**MFP3**

**(Specification 6360)**

**Further Pure 3**

**Final**

***Mark Scheme***

---

Mark schemes are prepared by the Principal Examiner and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all examiners participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every examiner understands and applies it in the same correct way. As preparation for standardisation each examiner analyses a number of students' scripts: alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, examiners encounter unusual answers which have not been raised they are required to refer these to the Principal Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this Mark Scheme are available from: [aqa.org.uk](http://aqa.org.uk)

Copyright © 2013 AQA and its licensors. All rights reserved.

**Copyright**

AQA retains the copyright on all its publications. However, registered schools/colleges for AQA are permitted to copy material from this booklet for their own internal use, with the following important exception: AQA cannot give permission to schools/colleges to photocopy any material that is acknowledged to a third party even for internal use within the centre.

Set and published by the Assessment and Qualifications Alliance.

## Key to mark scheme abbreviations

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
✓ or ft or F	follow through from previous incorrect result
CAO	correct answer only
CSO	correct solution only
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
A2,1	2 or 1 (or 0) accuracy marks
–x EE	deduct x marks for each error
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
c	candidate
sf	significant figure(s)
dp	decimal place(s)

## No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

**Otherwise we require evidence of a correct method for any marks to be awarded.**

Q	Solution	Marks	Total	Comments
1	$k_1 = 0.2 \times (2 - 1)\sqrt{2 + 1} \quad (= 0.2\sqrt{3})$ $= 0.346(410...) \quad (= *)$ $k_2 = 0.2 \times f(2.2, 1 + *...)$ $= 0.2 \times (2.2 - 1.346...) \sqrt{2.2 + 1.346...}$ $.... = 0.321(4946....)$ $y(2.2) = y(2) + \frac{1}{2}[k_1 + k_2]$ $= 1 + 0.5 \times [0.3464... + 0.3214...]$ $= 1 + 0.5 \times 0.667904....$ $(= 1.33395...) = 1.334 \text{ to 3dp}$	M1    M1  A1  m1  A1	5	PI. May be seen within given formula.  Accept 3dp or better as evidence of the M1 line.  $0.2 \times (2.2 - 1 - c's k_1) \sqrt{(2.2 + 1 + c's k_1)}$ PI May be seen within given formula.  3dp or better. PI by later work  Dep on previous two Ms but ft on c's numerical values for $k_1$ and $k_2$ following evaluation of these.  CAO Must be 1.334 SC Any <u>consistent</u> use of a MR/MC of printed $f(x,y)$ expression in applying IEF, mark as SC2 for a correct ft final 3dp value otherwise SC0.
	<b>Total</b>		<b>5</b>	
2	$(x + 8)^2 + (y - 6)^2 = 100$ $x^2 + y^2 + 16x - 12y + 64 + 36 (= 100)$  $r^2 + 16r \cos \theta - 12r \sin \theta = 0$  { $r=0$ , origin} Circle: $r = 12 \sin \theta - 16 \cos \theta$	B1   M1M1   A1	4	OE If polar form before expn of brackets award the B1 for correct expansions of both $(r \cos \theta - m)^2$ and $(r \sin \theta - n)^2$ where $(m,n) = (-8, 6)$ or $(m,n) = (6, -8)$  1 <sup>st</sup> M1 for replacement using any one of $\{[x^2 + y^2 = r^2, x = r \cos \theta, y = r \sin \theta](*)\}$  2 <sup>nd</sup> M1 for use of (*) to convert the form $x^2 + y^2 + ax + by = 0$ correctly to the form $r^2 + a \cos \theta + b \sin \theta = 0$ or better
	<b>Total</b>		<b>4</b>	

Q	Solution	Marks	Total	Comments
3(a)	$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 3y = 3x - 8e^{-3x}$ <p>P. Integral : <math>y_{PI} = a + bx + cxe^{-3x}</math></p> $y'_{PI} = b + ce^{-3x} - 3cxe^{-3x}$ $y''_{PI} = -6ce^{-3x} + 9cxe^{-3x}$ $-6ce^{-3x} + 9cxe^{-3x} + 2b + 2ce^{-3x} - 6cxe^{-3x}$ $-3a - 3bx - 3cxe^{-3x} = 3x - 8e^{-3x}$ $-3b = 3 ; 2b - 3a = 0; -4c = -8$ $b = -1 ; c = 2; a = -\frac{2}{3}$ $[ y_{PI} = -\frac{2}{3} - x + 2xe^{-3x} ]$	<p>M1</p> <p>M1</p> <p>m1</p> <p>A2,1,0</p>	5	<p>Product rule used at least once giving terms in the form <math>\pm pe^{-3x} \pm qxe^{-3x}</math></p> <p>Substitution into LHS of DE</p> <p>Dep on 2<sup>nd</sup> M only Equating coeffs to obtain at least two of these correct eqns; PI by correct values for at least two constants</p> <p>Dep on M1M1m1 all awarded A1 if any two correct; A2 if all three correct but do not award the 2<sup>nd</sup> A mark if terms in <math>xe^{-3x}</math> were incorrect in the M1 line</p>
(b)	<p>Aux. eqn. <math>m^2 + 2m - 3 = 0</math></p> $(m+3)(m-1) = 0$ $(y_{CF} =) Ae^{-3x} + Be^x$ $(y_{GS} =) Ae^{-3x} + Be^x - \frac{2}{3} - x + 2xe^{-3x}$	<p>M1</p> <p>A1</p> <p>B1F</p>	3	<p>Factorising or using quadratic formula OE PI by correct two values of 'm' seen/used</p> <p>c's CF + c's PI with 2 arbitrary constants, non-zero values for a,b and c and no trig or ln terms in c's CF</p>
(c)	$x = 0, y = 1 \Rightarrow 1 = A + B - \frac{2}{3}$ $\frac{dy}{dx} = -3Ae^{-3x} + Be^x - 1 + 2e^{-3x} - 6xe^{-3x}$ <p>As <math>x \rightarrow \infty, (e^{-3x} \rightarrow 0 \text{ and}) xe^{-3x} \rightarrow 0</math></p> <p>(As <math>x \rightarrow \infty, \frac{dy}{dx} \rightarrow -1</math> so) <math>B = 0</math></p> <p>When <math>B = 0, 1 = A - \frac{2}{3} \Rightarrow A = \frac{5}{3}</math></p> $y = \frac{5}{3}e^{-3x} - \frac{2}{3} - x + 2xe^{-3x}$	<p>B1F</p> <p>E1</p> <p>B1</p> <p>A1</p>	4	<p>Only fit if previous B1F has been awarded</p> <p>Must treat <math>xe^{-3x}</math> separately</p> <p><math>B=0</math>, where B is the coefficient of <math>e^x</math>.</p>
	<b>Total</b>		<b>12</b>	

Q	Solution	Marks	Total	Comments
4	$\int \left( \frac{2x}{x^2+4} - \frac{4}{2x+3} \right) dx = \ln(x^2+4) - 2\ln(2x+3) \{+c\}$ $(I =) \lim_{a \rightarrow \infty} \int_0^a \left( \frac{2x}{x^2+4} - \frac{4}{2x+3} \right) dx$ $= \lim_{a \rightarrow \infty} \left[ \ln(x^2+4) - 2\ln(2x+3) \right]_0^a$ $= \lim_{a \rightarrow \infty} \left[ \ln(a^2+4) - 2\ln(2a+3) \right] - (\ln 4 - 2\ln 3)$ $= \lim_{a \rightarrow \infty} \left[ \ln \left( \frac{a^2+4}{(2a+3)^2} \right) \right] - (\ln 4 - \ln 9)$ $= \lim_{a \rightarrow \infty} \left[ \ln \left( \frac{1 + \frac{4}{a^2}}{4 + \frac{12}{a} + \frac{9}{a^2}} \right) \right] - (\ln 4 - \ln 9)$ $I = \int_0^\infty \left( \frac{2x}{x^2+4} - \frac{4}{2x+3} \right) dx = \ln \frac{1}{4} - \ln \frac{4}{9} = \ln \frac{9}{16}$	<p>B1</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>A1</p>	<p>6</p>	<p>OE</p> <p><math>\infty</math> replaced by <math>a</math> (OE) and <math>\lim_{a \rightarrow \infty}</math> seen or taken at any stage</p> <p><b>Remaining marks are dep on</b> getting <math>p\ln(x^2+4)+q\ln(2x+3)</math> after integration, where <math>p</math> and <math>q</math> are non-zero constants</p> <p>Dealing with the 0 limit correctly and using <math>\ln P - \ln Q = \ln(P/Q)</math> at least once at any stage either before or after using <math>F(\cdot) - F(0)</math>. OE</p> <p>Writing <math>F(a)</math> OE in a suitable form when considering <math>a \rightarrow \infty</math>. OE</p> <p>CSO</p>
Total			6	

Q	Solution	Marks	Total	Comments
5(a)	$\frac{d}{dx}[\ln(\ln x)] = \frac{1}{\ln x} \times \frac{1}{x}$	B1	1	ACF
(b)(i)	$\frac{dy}{dx} + \frac{1}{x \ln x} y = 9x^2$ An IF is $\exp \left\{ \int [1/(x \ln x)] (dx) \right\}$ $= e^{\ln(\ln x)} = \ln x$	M1 A1	2	.... and with integration attempted AG Must see $e^{\ln(\ln x)}$ before $\ln x$
(ii)	$\ln x \frac{dy}{dx} + \frac{1}{x} y = 9x^2 \ln x$ $\frac{d}{dx}[y \ln x] = 9x^2 \ln x$ $y \ln x = \int 9x^2 \ln x dx$ $\Rightarrow y \ln x = \int \ln x d[3x^3]$ $= 3x^3 \ln x - \int 3x^3 \left( \frac{1}{x} \right) dx$ $y \ln x = 3x^3 \ln x - x^3 (+c)$ When $x = e$ , $y = 4e^3$ , $4e^3 = 3e^3 - e^3 + c$ $c = 2e^3$ $\Rightarrow y \ln x = 3x^3 \ln x - x^3 + 2e^3$ $y = 3x^3 - \frac{(x^3 - 2e^3)}{\ln x}$	M1 A1  m1 A1  m1 A1	6	LHS as differential of $y \times \ln x$ PI  $\int kx^2 \ln x (dx) = px^3 \ln x - \int px^3 \left( \frac{1}{x} \right) (dx)$ or better ACF Condone missing '+c' Dep on previous M1m1. Boundary condition used in attempt to find value of 'c' after integration is completed ACF
	<b>Total</b>		<b>9</b>	

Q	Solution	Marks	Total	Comments
6(a)	$y = (4 + \sin x)^{1/2}$ so $y^2 = 4 + \sin x$ $2y \frac{dy}{dx} = \cos x$ $y \frac{dy}{dx} = \frac{1}{2} \cos x$	M1 A1	2	$\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$
(a)	<b>Altn</b> $\frac{dy}{dx} = \frac{1}{2}(4 + \sin x)^{-1/2}(\cos x)$ $y \frac{dy}{dx} = \frac{1}{2} \cos x$	(M1) (A1)	(2)	Chain rule
(b)	$y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{1}{2} \sin x$ When $x = 0$ , $y = 2$ , $\frac{dy}{dx} = \frac{1}{4}$ , $2 \frac{d^2 y}{dx^2} + \left(\frac{1}{4}\right)^2 = 0$ $y \frac{d^3 y}{dx^3} + \frac{dy}{dx} \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} \frac{d^2 y}{dx^2} = -\frac{1}{2} \cos x$	M1 A1F m1 A1		Correct differentiation of $y \frac{dy}{dx}$ Ft on RHS of M1 line as $k \sin x$
	When $x=0$ , $2 \frac{d^3 y}{dx^3} + 3 \left(\frac{1}{4}\right) \left(-\frac{1}{32}\right) = -\frac{1}{2} \Rightarrow \frac{d^3 y}{dx^3} = -\frac{61}{256}$	A1	5	CSO
(b)	<b>Altn</b> $\frac{d^2 y}{dx^2} = -\frac{1}{4}(4 + \sin x)^{-3/2}(\cos^2 x) + \frac{1}{2}(4 + \sin x)^{-1/2}(-\sin x)$ $\frac{d^3 y}{dx^3} = \frac{3}{8}(4 + \sin x)^{-2.5}(\cos^3 x) - \frac{1}{4}(4 + \sin x)^{-1.5}(-2 \cos x \sin x)$ $- \frac{1}{4}(4 + \sin x)^{-1.5}(\cos x)(-\sin x) - \frac{1}{2}(4 + \sin x)^{-0.5} \cos x$	(M1) (A1) (m1) (A1)		Sign and numerical coeffs errors only. ACF Sign and numerical coeffs errors only. ACF
	When $x = 0$ , $\frac{d^3 y}{dx^3} = \frac{3}{8} \times \frac{1}{32} - \frac{1}{2} \times \left(\frac{1}{2}\right) = -\frac{61}{256}$	(A1)	(5)	CSO
(c)	McC. Thm: $y(0) + x y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{3!} y'''(0)$ $(4 + \sin x)^{1/2} \approx 2 + \frac{1}{4}x - \frac{1}{64}x^2 - \frac{61}{1536}x^3 \dots$	M1 A1		Maclaurin's theorem used with c's numerical values for $y(0)$ , $y'(0)$ , $y''(0)$ and $y'''(0)$ , all found with at least three being non-zero. CSO Previous 6 marks must have been scored
<b>Total</b>			<b>9</b>	



Q	Solution	Marks	Total	Comments
7(a)	$\sin^2 x \frac{d^2 y}{dx^2} - 2 \sin x \cos x \frac{dy}{dx} + 2y = 2 \sin^4 x \cos x$ $y = u \sin x$ $\frac{dy}{dx} = \frac{du}{dx} \sin x + u \cos x$ $\frac{d^2 y}{dx^2} = \frac{d^2 u}{dx^2} \sin x + \frac{du}{dx} \cos x + \frac{du}{dx} \cos x - u \sin x$ $\frac{d^2 u}{dx^2} \sin^3 x + 2 \frac{du}{dx} \cos x \sin^2 x - u \sin^3 x - 2 \frac{du}{dx} \sin^2 x \cos x - 2u \sin x \cos^2 x + 2u \sin x = 2 \sin^4 x \cos x$ $\frac{d^2 u}{dx^2} \sin^3 x + u \sin x [-\sin^2 x - 2 \cos^2 x + 2] = 2 \sin^4 x \cos x$ $\frac{d^2 u}{dx^2} \sin^3 x + u \sin x [-\sin^2 x + 2 \sin^2 x] = 2 \sin^4 x \cos x$ <p>(Divide throughout by <math>\sin^3 x</math> ,) <math>\frac{d^2 u}{dx^2} + u = 2 \sin x \cos x</math></p> $\Rightarrow \frac{d^2 u}{dx^2} + u = \sin 2x$	<p>M1</p> <p>A1</p> <p>m1</p> <p>A1</p> <p>A1</p>	<p>5</p> <p>6</p>	<p>Both derivatives attempted and product rule used at least twice.</p> <p>Both correct</p> <p>Substitution into original DE</p> <p>Need to see clear use of the trig identity</p> <p>AG Completion, be convinced</p> <p>PI</p> <p>OE</p> <p>Condone extra terms provided their coefficients are shown to be zero</p> <p>Correct Particular integral</p> <p><math>u=g(x)</math>, where <math>g(x)=c</math>'s (CF+PI) with two arb. constants, <math>PI \neq 0</math> and all real. Can be implied by next line.</p> <p><math>y=f(x)</math> with ACF for <math>f(x)</math></p>
(b)	<p>For <math>\frac{d^2 u}{dx^2} + u = \sin 2x</math> , aux eqn, <math>m^2+1=0 \Rightarrow m = \pm i</math></p> <p>CF: <math>(u = ) A \sin x + B \cos x</math></p> <p>For PI try <math>(u = ) p \sin 2x</math></p> $-4p \sin 2x + p \sin 2x = \sin 2x \Rightarrow p = -\frac{1}{3}$ <p>GS for <math>u = A \sin x + B \cos x - \frac{1}{3} \sin 2x</math></p> <p>GS: <math>y = A \sin^2 x + B \sin x \cos x - \frac{1}{3} \sin 2x \sin x</math></p>	<p>M1</p> <p>A1</p> <p>M1</p> <p>A1</p> <p>B1F</p> <p>A1</p>	<p>6</p>	
	<b>Total</b>		<b>11</b>	

Q	Solution	Marks	Total	Comments
8(a)	At intersections of $r=2$ and $r = 3 + 2 \sin \theta$ $2 = 3 + 2 \sin \theta$ $\sin \theta = -\frac{1}{2}, \Rightarrow \theta = \frac{7}{6}\pi, \theta = \frac{11}{6}\pi$ $(P=) \left(2, \frac{7\pi}{6}\right), (Q=) \left(2, \frac{11\pi}{6}\right)$	M1 A1 A1	3	Elimination of $r$ Any one correct solution of $\sin \theta = -\frac{1}{2}$ $\left(2, \frac{7\pi}{6}\right)$ and $\left(2, \frac{11\pi}{6}\right)$
(b)(i)	Angle between $OA$ and initial line $= \frac{\pi}{6}$  When $\theta = \frac{\pi}{6}, r = 3 + 2 \sin \frac{\pi}{6} = 4;$ $A\left(4, \frac{\pi}{6}\right)$	B1F  B1	2	If not correct, ft on $\theta_p - \pi$
(ii)	$OA = 4, OQ = 2$ Angle $AOQ = \pi - (\theta_Q - \theta_P) = \frac{\pi}{3}$  $AQ^2 = 4^2 + 2^2 - 2(4)(2) \cos AOQ (=12)$ $AQ = \sqrt{12}$	B1F  M1 A1	3	If not correct, ft on $\pi - (\theta_Q - \theta_P)$ . OE eg Cartesian coords of $A$ and $Q$ both attempted and at least one correct ft. Valid method to find $AQ$ (or $AQ^2$ ). Ft on c's $r_A$ for $OA$ ACF but must be exact surd form.
(iii)	Since $4^2 = 2^2 + (\sqrt{12})^2$ so $90^\circ$ angle $OQA = 90^\circ \Rightarrow AQ$ is a tangent	E1 E1	2	Justifying why (angle $OQA = 90^\circ$ ) OE Must have convincingly shown that $OQA = 90^\circ$
(c)	Area of minor sector $OPQ$ of circle $= \frac{1}{2}(2)^2 [\theta_Q - \theta_P]$ $= \frac{4\pi}{3}$ Area of minor region $OPQ$ of curve $=$ $\frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (4 \sin^2 \theta + 12 \sin \theta + 9) d\theta$  $= \frac{1}{2} \int (2 - 2 \cos 2\theta + 12 \sin \theta + 9) d\theta$  $= \frac{1}{2} [2\theta - \sin 2\theta - 12 \cos \theta + 9\theta] =$ $\left[ \frac{121\pi}{12} + \frac{\sqrt{3}}{4} - \frac{6\sqrt{3}}{2} \right] - \left[ \frac{77\pi}{12} - \frac{\sqrt{3}}{4} + \frac{6\sqrt{3}}{2} \right]$ $\{ = \frac{11\pi}{3} - \frac{11\sqrt{3}}{2} \}$  Area of shaded region $= \frac{4\pi}{3} - \{ \frac{11\pi}{3} - \frac{11\sqrt{3}}{2} \}$  $= \frac{11}{2} \sqrt{3} - \frac{7}{3} \pi = \frac{1}{6} (33\sqrt{3} - 14\pi)$	M1 A1  M1 B1 M1 A1F A1 M1 A1	9	$\frac{1}{2}(2)^2 [\theta_Q - \theta_P]$ PI by combined $-\frac{7\pi}{3}$ OE term later.  Use of $\frac{1}{2} \int r^2 d\theta$ or use of $\int_{\theta_P}^{3\pi/2} r^2 d\theta$ OE $r^2 = 4 \sin^2 \theta + 12 \sin \theta + 9$ Use of $\cos 2\theta = \pm 1 \pm 2 \sin^2 \theta$ with $k \int r^2 (d\theta)$ Ft wrong non zero coefficients, ie for correct integration of $a + b \cos 2\theta + c \sin \theta$ OE eg $\left[ \frac{33\pi}{2} \right] - \left[ \frac{77\pi}{6} - \frac{\sqrt{3}}{2} + 6\sqrt{3} \right]$ eg $\left[ \frac{121\pi}{6} + \frac{\sqrt{3}}{2} - 6\sqrt{3} \right] - \left[ \frac{33\pi}{2} \right]$ $\frac{1}{2}(2)^2 [\theta_Q - \theta_P] - \frac{1}{2} \int_{\theta_P}^{\theta_Q} (3 + 2 \sin \theta)^2 d\theta$ CSO $\frac{1}{6} (33\sqrt{3} - 14\pi)$ . ( $m = 33, n = -14$ )
	<b>Total</b>		<b>19</b>	
	<b>TOTAL</b>		<b>75</b>	