4727 Further Pure Mathematics 3

1	METHOD 1			
	line segment between l_1 and l_2 = $\pm[4, -3, -9]$	B1		For correct vector
	$\mathbf{n} = [1, -1, 2] \times [2, 3, 4] = (\pm)[-2, 0, 1]$	M1* A1		For finding vector product of direction vectors
	distance = $\dfrac{\lvert[4, -3, -9] . [-2, 0, 1]\rvert}{\left(\sqrt{2^2+0^2+1^2}\right)} = \dfrac{17}{\left(\sqrt{5}\right)}$	M1 (*dep)		For using numerator of distance formula
	$\neq 0$, so skew	A1	**5**	For correct scalar product and correct conclusion
	METHOD 2 lines would intersect where			
	$\left.\begin{aligned} 1+s &= -3+2t \\ -2-s &= 1+3t \\ -4+2s &= 5+4t \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} s-2t &= -4 \\ s+3t &= -3 \\ 2s-4t &= 9 \end{aligned}\right.$	B1 M1* A1		For correct parametric form for either line For 3 equations using 2 different parameters
		M1 (*dep)		For attempting to solve to show (in)consistency
	$\Rightarrow$ contradiction, so skew	A1		For correct conclusion
			$\boxed{5}$	
2 (i)	$\left(a+b\sqrt{5}\right)\left(c+d\sqrt{5}\right)$	M1		For using product of 2 distinct elements
	$= ac+5bd+(bc+ad)\sqrt{5} \in H$	A1	**2**	For correct expression
(ii)	$(e=)$ 1 *OR* $1+0\sqrt{5}$	B1	**1**	For correct identity
(iii)	*EITHER* $\dfrac{1}{a+b\sqrt{5}} \times \dfrac{a-b\sqrt{5}}{a-b\sqrt{5}}$	M1		For correct inverse as $\left(a+b\sqrt{5}\right)^{-1}$ and multiplying top and bottom by $a-b\sqrt{5}$
	OR $\left(a+b\sqrt{5}\right)\left(c+d\sqrt{5}\right)=1 \Rightarrow \left\{\begin{aligned} ac+5bd &= 1 \\ bc+ad &= 0 \end{aligned}\right.$			*OR* for using definition and equating parts
	inverse $= \dfrac{a}{a^2-5b^2} - \dfrac{b}{a^2-5b^2}\sqrt{5}$	A1	**2**	For correct inverse. Allow as a single fraction
(iv)	5 is prime *OR* $\sqrt{5} \notin \mathbb{Q}$	B1	**1**	For a correct property (or equivalent)
			$\boxed{6}$	
3	Integrating factor = $e^{\int 2\,dx} = e^{2x}$	B1		For correct IF
	$\Rightarrow \dfrac{d}{dx}\left(ye^{2x}\right) = e^{-x}$	M1		For $\dfrac{d}{dx}(y.\text{their IF}) = e^{-3x}$. their IF
	$\Rightarrow y\,e^{2x} = -e^{-x}(+c)$	A1		For correct integration both sides
	$(0, 1) \Rightarrow c = 2$	M1		For substituting $(0, 1)$ into their GS and solving for c
		A1√		For correct c f.t. from their GS
	$\Rightarrow y = -e^{-3x} + 2e^{-2x}$	A1	**6**	For correct solution
			$\boxed{6}$	
4 (i)	$(z=)$ $2, -2, 2i, -2i$	M1		For at least 2 roots of the form $k\{1, i\}$ **AEF**
		A1	**2**	For correct values

(ii)	$\frac{w}{1-w} = 2, -2,\ 2\mathrm{i}, -2\mathrm{i}$	M1		For $\frac{w}{1-w}$ = any one solution from **(i)**
	$w = \frac{z}{1+z}$	M1		For attempting to solve for w, using any solution or in general
	$w = \frac{2}{3}, 2$	B1 A1		For any one of the 4 solutions For both real solutions
	$w = \frac{4}{5} \pm \frac{2}{5}\mathrm{i}$	A1	**5**	For both complex solutions **SR** Allow B1√ and one A1√ from $k \neq 2$
			$\boxed{7}$	
5 (i)	$\mathbf{AB} = k\left[\frac{2}{3}\sqrt{3}, 0, -\frac{2}{3}\sqrt{6}\right]$, $\mathbf{BC} = k\left[-\sqrt{3}, 1, 0\right]$, $\mathbf{CA} = k\left[\frac{1}{3}\sqrt{3}, -1, \frac{2}{3}\sqrt{6}\right]$	B1 B1		For any one edge vector of ΔABC For any other edge vector of ΔABC
	$\mathbf{n} = k_1\left[\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{18}, \frac{2}{3}\sqrt{3}\right] = k_2\left[1, \sqrt{3}, \frac{1}{2}\sqrt{2}\right]$	M1		For attempting to find vector product of any two edges
		M1		For substituting A, B or C into $\mathbf{r}.\mathbf{n}$
	substitute A, B or C $\Rightarrow$ $x + \sqrt{3}y + \frac{1}{2}\sqrt{2}z = \frac{2}{3}\sqrt{3}$	A1	**5**	For correct equation **AG**
				SR For verification only allow M1, then A1 for 2 points and A1 for the third point
(ii)	Symmetry in plane OAB or Oxz or $y = 0$	B1* B1 (*dep)	**2**	For quoting symmetry or reflection For correct plane Allow "in y coordinates" or "in y axis" **SR** For symmetry implied by reference to opposite signs in y coordinates of C and D, award B1 only
(iii)	$\cos\theta = \frac{\left\|\left[1, \sqrt{3}, \frac{1}{2}\sqrt{2}\right].\left[1, -\sqrt{3}, \frac{1}{2}\sqrt{2}\right]\right\|}{\sqrt{1+3+\frac{1}{2}}\sqrt{1+3+\frac{1}{2}}}$	M1		For using scalar product of normal vectors
		A1		For correct scalar product
	$= \frac{\left\|1-3+\frac{1}{2}\right\|}{\frac{9}{2}} = \frac{\frac{3}{2}}{\frac{9}{2}} = \frac{1}{3}$	M1		For product of both moduli in denominator
		A1	**4**	For correct answer. Allow $-\frac{1}{3}$
			$\boxed{11}$	
6 (i)	$\left(m^2 + 16 = 0 \Rightarrow\right)$ $m = \pm 4\mathrm{i}$	M1		For attempt to solve correct auxiliary equation (may be implied by correct CF)
	CF = $A\cos 4x + B\sin 4x$	A1	**2**	For correct CF (**AEtrig** but not $A\mathrm{e}^{4\mathrm{i}x} + B\mathrm{e}^{-4\mathrm{i}x}$ only)
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = p\sin 4x + 4px\cos 4x$	M1		For differentiating PI twice, using product rule
		A1		For correct $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 8p\cos 4x - 16px\sin 4x$	A1√		For unsimplified $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. f.t. from $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$\Rightarrow 8p\cos 4x = 8\cos 4x$	M1		For substituting into DE
	$\Rightarrow p = 1$	A1		For correct p
	$\Rightarrow (y =) A\cos 4x + B\sin 4x + x\sin 4x$	B1√	**6**	For using GS = CF + PI, with 2 arbitrary constants in CF and none in PI

(iii)	$(0, 2) \Rightarrow A = 2$	B1√		For correct A. f.t. from their GS
	$\frac{dy}{dx} = -4A\sin 4x + 4B\cos 4x + \sin 4x + 4x\cos 4x$	M1		For differentiating their GS
	$x = 0,\ \frac{dy}{dx} = 0 \Rightarrow B = 0$	M1		For substituting values for x and $\frac{dy}{dx}$ to find B
	$\Rightarrow y = 2\cos 4x + x\sin 4x$	A1	**4**	For stating correct solution **CAO** including $\boxed{y=}$
			$\boxed{12}$	
7 (i)	$\cos 6\theta = 0 \Rightarrow 6\theta = k \times \frac{1}{2}\pi$	M1		For multiples of $\frac{1}{2}\pi$ seen or implied
	$\Rightarrow \theta = \frac{1}{12}\pi\{1, 3, 5, 7, 9, 11\}$	A1		A1 for any 3 correct
		A1	**3**	A1 for the rest, and no extras in $0 < \theta < \pi$
(ii)	METHOD 1			
	$\mathrm{Re}(c + \mathrm{i}\,s)^6 = \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	M1		For expanding $(c + \mathrm{i}\,s)^6$ at least 4 terms and 2 binomial coefficients needed
		A1		For 4 correct terms
	$\cos 6\theta = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$	M1		For using $s^2 = 1 - c^2$
	$\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$	A1		For correct expression for $\cos 6\theta$
	$\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$	A1	**5**	For correct result **AG** (may be written down from correct $\cos 6\theta$)
	METHOD 2			
	$\mathrm{Re}(c + \mathrm{i}\,s)^3 = \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$	M1		For expanding $(c + \mathrm{i}\,s)^3$ at least 2 terms and 1 binomial coefficient needed
		A1		For 2 correct terms
	$\Rightarrow \cos 6\theta = \cos 2\theta\left(\cos^2 2\theta - 3\sin^2 2\theta\right)$	M1		For replacing θ by 2θ
	$\Rightarrow \cos 6\theta = \left(2\cos^2\theta - 1\right)\left(4\left(2\cos^2\theta - 1\right)^2 - 3\right)$	A1		For correct expression in $\cos\theta$ (unsimplified)
	$\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$	A1		For correct result **AG**
(iii)	METHOD 1			
	$\cos 6\theta = 0$	M1		For putting $\cos 6\theta = 0$
	$\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $16c^4 - 16c^2 + 1 = 0$ and $2c^2 - 1 = 0$	A1		For association of roots with quartic and quadratic
	But $\theta = \frac{1}{4}\pi, \frac{3}{4}\pi$ satisfy $2c^2 - 1 = 0$	B1		For correct association of roots with quadratic
	EITHER Product of 4 roots *OR* $c = \pm\frac{1}{2}\sqrt{2 \pm \sqrt{3}}$	M1		For using product of 4 roots *OR* for solving quartic
	$\Rightarrow \cos\frac{1}{12}\pi\cos\frac{5}{12}\pi\cos\frac{7}{12}\pi\cos\frac{11}{12}\pi = \frac{1}{16}$	A1	**5**	For correct value (may follow A0 and B0)

	METHOD 2		
	$\cos 6\theta = 0$	M1	For putting $\cos 6\theta = 0$
	$\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $32c^6 - 48c^4 + 18c^2 - 1 = 0$	A1	For association of roots with sextic
	Product of 6 roots $\Rightarrow$	M1	For using product of 6 roots
	$\cos\frac{1}{12}\pi \cdot \frac{1}{\sqrt{2}} \cdot \cos\frac{5}{12}\pi \cos\frac{7}{12}\pi \cdot \frac{-1}{\sqrt{2}} \cdot \cos\frac{11}{12}\pi = -\frac{1}{32}$	B1	For using $\cos\left\{\frac{3}{12}\pi, \frac{9}{12}\pi\right\} = \left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\}$
	$\cos\frac{1}{12}\pi \cos\frac{5}{12}\pi \cos\frac{7}{12}\pi \cos\frac{11}{12}\pi = \frac{1}{16}$	A1	For correct value
		13	
8 (i)	$g(x) = \dfrac{1}{2 - 2 \cdot \dfrac{1}{2-2x}} = \dfrac{2-2x}{2-4x} = \dfrac{1-x}{1-2x}$	M1 A1	For use of $ff(x)$ For correct expression **AG**
	$gg(x) = \dfrac{1 - \dfrac{1-x}{1-2x}}{1 - 2 \cdot \dfrac{1-x}{1-2x}} = \dfrac{-x}{-1} = x$	M1 A1 **4**	For use of $gg(x)$ For correct expression **AG**
(ii)	Order of f = 4 order of g = 2	B1 B1 **2**	For correct order For correct order
(iii)	METHOD 1		
	$y = \dfrac{1}{2-2x} \Rightarrow x = \dfrac{2y-1}{2y}$	M1	For attempt to find inverse
	$\Rightarrow f^{-1}(x) = h(x) = \dfrac{2x-1}{2x}$ *OR* $1 - \dfrac{1}{2x}$	A1 **2**	For correct expression
	METHOD 2		
	$f^{-1} = f^3 = fg$ or gf	M1	For use of $fg(x)$ or $gf(x)$
	$fg(x) = h(x) = \dfrac{1}{2 - 2\left(\dfrac{1-x}{1-2x}\right)} = \dfrac{1-2x}{-2x}$	A1	For correct expression
(iv)	$\begin{array}{c\|cccc} & e & f & g & h \\ \hline e & e & f & g & h \\ f & f & g & h & e \\ g & g & h & e & f \\ h & h & e & f & g \end{array}$	M1 A1 A1 A1 **4**	For correct row 1 and column 1 For e, f, g, h in a latin square For correct diagonal e - g - e - g For correct table
		12	