4727 Further Pure Mathematics 3

1		$\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\mathrm{i}\right)^{\frac{1}{3}} = \left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)^{\frac{1}{3}}$	B1		For $\arg z = \frac{1}{6}\pi$ seen or implied
		$= \cos\frac{1}{18}\pi + \mathrm{i}\sin\frac{1}{18}\pi$,	M1		For dividing $\arg z$ by 3
		$\cos\frac{13}{18}\pi + \mathrm{i}\sin\frac{13}{18}\pi$,	A1		For any one correct root
		$\cos\frac{25}{18}\pi + \mathrm{i}\sin\frac{25}{18}\pi$	A1	**4**	For 2 other roots and no more in range $0 \leqslant \theta < 2\pi$
			[4]		
2	**(i)**	$\frac{1}{5}\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$	B1	**1**	For stating correct inverse in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$
	(ii)	$r_1\mathrm{e}^{\mathrm{i}\theta} \times r_2\mathrm{e}^{\mathrm{i}\phi} = r_1 r_2\,\mathrm{e}^{\mathrm{i}(\theta+\phi)}$	M1 A1	 **2**	For stating 2 distinct elements multiplied For showing product of correct form
	(iii)	$Z^2 = \mathrm{e}^{2\mathrm{i}\gamma}$	B1		For $\mathrm{e}^{2\mathrm{i}\gamma}$ seen or implied
		$\Rightarrow\ \mathrm{e}^{2\mathrm{i}\gamma - 2\pi\mathrm{i}}$	B1	**2**	For correct answer. aef
			[5]		
3	**(i)**	$[6-4\lambda, -7+8\lambda, -10+7\lambda]$ on p	B1		For point on l seen or implied
		$\Rightarrow 3(6-4\lambda) - 4(-7+8\lambda) - 2(-10+7\lambda) = 8$	M1		For substituting into equation of p
		$\Rightarrow\ \lambda = 1 \Rightarrow\ (2, 1, -3)$	A1	**3**	For correct point. Allow position vector
	(ii)	METHOD 1			
		$\mathbf{n} = [-4, 8, 7] \times [3, -4, -2]$	M1* M1 (*dep)		For direction of l and normal of p seen For attempting to find $\mathbf{n}_1 \times \mathbf{n}_2$
		$\mathbf{n} = k[12, 13, -8]$	A1		For correct vector
		$(2, 1, -3)$ OR $(6, -7, -10)$	M1		For finding scalar product of their point on l with their attempt at $\mathbf{n}$, or equivalent
		$\Rightarrow\ 12x + 13y - 8z = 61$	A1	**5**	For correct equation, aef cartesian
		METHOD 2			
		$\mathbf{r} = [2, 1, -3]$ *OR* $[6, -7, -10]$ $+\lambda[-4, 8, 7] + \mu[3, -4, -2]$	M1 A1√		For stating eqtn of plane in parametric form (may be implied by next stage), using $[2, 1, -3]$ (ft from **(i)**) Or $[6, -7, -10]$, $\mathbf{n}_1$ and $\mathbf{n}_2$ (as above)
		$x = 2 - 4\lambda + 3\mu$ $y = 1 + 8\lambda - 4\mu$ $z = -3 + 7\lambda - 2\mu$	M1 M1		For writing as 3 linear equations For attempting to eliminate λ and μ
		$\Rightarrow\ 12x + 13y - 8z = 61$	A1		For correct equation aef cartesian
		METHOD 3			
		$3(6+3\mu) - 4(-7-4\mu) - 2(-10-2\mu) = 8$	M1		For finding foot of perpendicular from point on l to p
		$\Rightarrow\ \mu = -2 \Rightarrow\ (0, 1, -6)$	A1		For correct point or position vector
		From 3 points $(2, 1, -3)$, $(6, -7, -10)$, $(0, 1, -6)$, $\mathbf{n}$ = vector product of 2 of $[2, 0, 3]$, $[6, -8, -4]$, $[-4, 8, 7]$	M1		Use vector product of 2 vectors in plane
		$\Rightarrow\ \mathbf{n} = k[12, 13, -8]$			
		$(2, 1, -3)$ OR $(6, -7, -10)$	M1		For finding scalar product of their point on l with their attempt at $\mathbf{n}$, or equivalent
		$\Rightarrow\ 12x + 13y - 8z = 61$	A1		For correct equation aef cartesian
			[8]		

4	**(i)**	IF $e^{\int \frac{1}{1-x^2}dx} = e^{\frac{1}{2}\ln\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$	M1 A1 **2**	For IF stated or implied. Allow $\pm\int$ and omission of dx For integration and simplification to **AG** (intermediate step must be seen)
	(ii)	$\frac{d}{dx}\left(y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right) = (1+x)^{\frac{1}{2}}$	M1*	For multiplying both sides by IF
		$y\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{2}{3}(1+x)^{\frac{3}{2}} + c$	M1 A1	For integrating RHS to $k(1+x)^n$ For correct equation (including + c)
		$(0,\ 2) \Rightarrow\ 2 = \frac{2}{3} + c\ \Rightarrow\ c = \frac{4}{3}$	M1 (*dep) M1 (*dep)	In either order: For substituting $(0,\ 2)$ into their GS (including +c) For dividing solution through by IF, including dividing c or their numerical value for c
		$y = \frac{2}{3}(1+x)(1-x)^{\frac{1}{2}} + \frac{4}{3}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$	A1 **6**	For correct solution aef (even unsimplified) in form $y = f(x)$
			8	
5	**(i)**	$m^2 - 6m + 9\ (=0) \Rightarrow\ m = 3$	M1 A1	For attempting to solve correct auxiliary equation For correct m
		CF = $(A + Bx)e^{3x}$	A1 **3**	For correct CF
	(ii)	ke^{3x} and kxe^{3x} both appear in CF	B1 **1**	For correct statement
	(iii)	$y = kx^2e^{3x}\ \Rightarrow\ y' = 2kxe^{3x} + 3kx^2e^{3x}$	M1 A1	For differentiating kx^2e^{3x} twice For correct y' aef
		$\Rightarrow\ y'' = 2ke^{3x} + 12kxe^{3x} + 9kx^2e^{3x}$	A1	For correct y'' aef
		$\Rightarrow$ $ke^{3x}\left(2 + 12x + 9x^2 - 12x - 18x^2 + 9x^2\right) = e^{3x}$	M1	For substituting $y'',\ y',\ y$ into DE
		$\Rightarrow\ k = \frac{1}{2}$	A1 **5**	For correct k
			9	

6 (i)	METHOD 1			
	$\mathbf{n}_1 = [1, 1, 0] \times [1, -5, -2]$	M1		For attempting to find vector product of the pair of direction vectors
	$= [-2, 2, -6] = k[1, -1, 3]$	A1		For correct $\mathbf{n}_1$
	Use $(2, 2, 1)$	M1		For substituting a point into equation
	$\Rightarrow \mathbf{r}.[-2, 2, -6] = -6 \Rightarrow \mathbf{r}.[1, -1, 3] = 3$	A1	**4**	For correct equation. aef in this form
	METHOD 2			
	$x = 2 + \lambda + \mu$, $y = 2 + \lambda - 5\mu$, $z = 1 \quad - 2\mu$	M1 M1		For writing as 3 linear equations For attempting to eliminate λ and μ
	$\Rightarrow x - y + 3z = 3$	A1		For correct cartesian equation
	$\Rightarrow \mathbf{r}.[1, -1, 3] = 3$	A1		For correct equation. aef in this form
(ii)	For $\mathbf{r} = \mathbf{a} + t\mathbf{b}$			
	METHOD 1			
	$\mathbf{b} = [1, -1, 3] \times [7, 17, -3]$	M1		For attempting to find $\mathbf{n}_1 \times \mathbf{n}_2$
	$= k[2, -1, -1]$	A1√		For a correct vector. ft from $\mathbf{n}_1$ in **(i)**
	e.g. x, y or $z = 0$ in $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$	M1		For attempting to find a point on the line
	$\Rightarrow \mathbf{a} = \left[0, \frac{3}{2}, \frac{3}{2}\right]$ OR $[3, 0, 0]$ OR $[1, 1, 1]$	A1√		For a correct vector. ft from equation in **(i)** **SR** a correct vector may be stated without working
	Line is (e.g.) $\mathbf{r} = [1, 1, 1] + t[2, -1, -1]$	A1√	**5**	For stating equation of line ft from **a** and **b** **SR** for $\mathbf{a} = [2, 2, 1]$ stated award M0
	METHOD 2			
	Solve $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$ by eliminating one variable (e.g. z)	M1		In either order: For attempting to solve equations
	Use parameter for another variable (e.g. x) to find other variables in terms of t	M1		For attempting to find parametric solution
	(eg) $y = \frac{3}{2} - \frac{1}{2}t$, $z = \frac{3}{2} - \frac{1}{2}t$	A1√ A1√		For correct expression for one variable For correct expression for the other variable ft from equation in **(i)** for both
	Line is (eg) $\mathbf{r} = \left[0, \frac{3}{2}, \frac{3}{2}\right] + t[2, -1, -1]$	A1√		For stating equation of line. ft from parametric solutions
	METHOD 3			
	eg x, y or $z = 0$ in $\begin{cases} x - y + 3z = 3 \\ 7x + 17y - 3z = 21 \end{cases}$	M1		For attempting to find a point on the line
	$\Rightarrow \mathbf{a} = \left[0, \frac{3}{2}, \frac{3}{2}\right]$ *OR* $[3, 0, 0]$ *OR* $[1, 1, 1]$	A1√		For a correct vector. ft from equation in **(i)** **SR** a correct vector may be stated without working **SR** for $\mathbf{a} = [2, 2, 1]$ stated award M0
	eg $[3, 0, 0] - [1, 1, 1]$	M1		For finding another point on the line and using it with the one already found to find **b**
	$\mathbf{b} = k[2, -1, -1]$	A1√		For a correct vector. ft from equation in **(i)**
	Line is (eg) $\mathbf{r} = [1, 1, 1] + t[2, -1, -1]$	A1√		For stating equation of line. ft from **a** and **b**

6 (ii) contd	METHOD 4			
	A point on Π_1 is $[2+\lambda+\mu,\ 2+\lambda-5\mu,\ 1-2\mu]$	M1		For using parametric form for Π_1 and substituting into Π_2
	On $\Pi_2 \Rightarrow$ $[2+\lambda+\mu,\ 2+\lambda-5\mu,\ 1-2\mu].[7, 17, -3]=21$	A1		For correct unsimplified equation
	$\Rightarrow\ \lambda-3\mu=-1$	A1		For correct equation
	Line is (e.g.) $\mathbf{r}=[2, 2, 1]+(3\mu-1)[1, 1, 0]+\mu[1, -5, -2]$	M1		For substituting into Π_1 for λ *or* μ
	$\Rightarrow\ \mathbf{r}=[1,1,1]$ *or* $\left[\frac{7}{3}, \frac{1}{3}, \frac{1}{3}\right]+t\,[2, -1, -1]$	A1		For stating equation of line
			9	
7 (i)	$\cos 3\theta+\mathrm{i}\sin 3\theta=c^3+3\mathrm{i}\,c^2s-3cs^2-\mathrm{i}\,s^3$	M1		For using de Moivre with $n=3$
	$\Rightarrow \cos 3\theta=c^3-3cs^2$ and $\sin 3\theta=3c^2s-s^3$	A1		For both expressions in this form (seen or implied) **SR** For expressions found without de Moivre M0 A0
	$\Rightarrow \tan 3\theta=\dfrac{3c^2s-s^3}{c^3-3cs^2}$	M1		For expressing $\dfrac{\sin 3\theta}{\cos 3\theta}$ in terms of c and s
	$\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\dfrac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$	A1	**4**	For simplifying to **AG**
(ii) (a)	$\theta=\frac{1}{12}\pi \Rightarrow\ \tan 3\theta=1$ $\Rightarrow\ 1-3t^2=t(3-t^2)\ \Rightarrow$ $t^3-3t^2-3t+1=0$	B1	**1**	For both stages correct **AG**
(b)	$(t+1)(t^2-4t+1)=0$	M1 A1		For attempt to factorise cubic For correct factors
	$\Rightarrow\ (t=-1),\ \ t=2\pm\sqrt{3}$	A1		For correct roots of quadratic
	– sign for smaller root $\Rightarrow$ $\tan\frac{1}{12}\pi=2-\sqrt{3}$	A1	**4**	For choice of – sign and correct root **AG**
(iii)	$\mathrm{d}t=(1+t^2)\,\mathrm{d}\theta$	B1		For differentiation of substitution and use of $\sec^2\theta=1+\tan^2\theta$
	$\Rightarrow \int_0^{\frac{1}{12}\pi}\tan 3\theta\ \mathrm{d}\theta$	B1		For integral with correct θ limits seen
	$=\left[\frac{1}{3}\ln\left(\sec 3\theta\right)\right]_0^{\frac{1}{12}\pi}=\frac{1}{3}\ln\left(\sec\frac{1}{4}\pi\right)$	M1		For integrating to $k\ln\left(\sec 3\theta\right)$ OR $k\ln\left(\cos 3\theta\right)$
	$=\frac{1}{3}\ln\sqrt{2}=\frac{1}{6}\ln 2$	M1		For substituting limits and $\sec\frac{1}{4}\pi=\sqrt{2}$ OR $\cos\frac{1}{4}\pi=\frac{1}{\sqrt{2}}$ seen
		A1	**5**	For correct answer aef
			14	

8	**(i)**	$a^2 = (ap)^2 = apap \Rightarrow a = pap$	B1		For use of given properties to obtain **AG**
		$p^2 = (ap)^2 = apap \Rightarrow p = apa$	B1	**2**	For use of given properties to obtain **AG** **SR** allow working from **AG** to obtain relevant properties
	(ii)	$\left(p^2\right)^2 = p^4 = e \Rightarrow$ order $p^2 = 2$	B1		For correct order with no incorrect working seen
		$\left(a^2\right)^2 = \left(p^2\right)^2 = e \Rightarrow$ order $a = 4$	B1		For correct order with no incorrect working seen
		$(ap)^4 = a^4 = e \Rightarrow$ order $ap = 4$	B1		For correct order with no incorrect working seen
		$\left(ap^2\right)^2 = ap^2ap^2 = ap\,.\,a\,.\,p = a^2$ *OR* $ap^2 = a.a^2 = a^3 \Rightarrow$ $\left(ap^2\right)^2 = a^6 = a^2$ $\Rightarrow$ order $ap^2 = 4$	M1 A1	 **5**	For relevant use of **(i)** or given properties For correct order with no incorrect working seen
	(iii)	METHOD 1 $p^2 = a^2,\ ap^2 = a^3$	M2		For use of the given properties to simplify p^2 and $a\,p^2$
		$\Rightarrow \{e, a, p^2, ap^2\} = \{e, a, a^2, a^3\}$	A1		For obtaining a^2 and a^3
		which is a cyclic group	A1	**4**	For justifying that the set is a group
		METHOD 2 $\begin{array}{c\|cccc} & e & a & p^2 & ap^2 \\ \hline e & e & a & p^2 & ap^2 \\ a & a & p^2 & ap^2 & e \\ p^2 & p^2 & ap^2 & e & a \\ ap^2 & ap^2 & e & a & p^2 \end{array}$	M1 A1		For attempting closure with all 9 non-trivial products seen For all 16 products correct
		Completed table is a cyclic group	B2		For justifying that the set is a group
		METHOD 3 $\begin{array}{c\|cccc} & e & a & p^2 & ap^2 \\ \hline e & e & a & p^2 & ap^2 \\ a & a & p^2 & ap^2 & e \\ p^2 & p^2 & ap^2 & e & a \\ ap^2 & ap^2 & e & a & p^2 \end{array}$	M1 A1		For attempting closure with all 9 non-trivial products seen For all 16 products correct
		Identity = e	B1		For stating identity
		Inverses exist since EITHER: e is in each row/column OR: p^2 is self-inverse; a, ap^2 form an inverse pair	B1		For justifying inverses ($e^{-1} = e$ may be assumed)

(iv)	METHOD 1 e.g. $\left.\begin{array}{l} a\,.\,ap = a^2 p = p^3 \\ ap\,.\,a = p \end{array}\right\} \Rightarrow$ not commutative	M1 M1 B1 A1	 **4**	For attempting to find a non-commutative pair of elements, at least one involving *a* (may be embedded in a full or partial table) For simplifying elements both ways round For a correct pair of non-commutative elements For stating *Q* non-commutative, with a clear argument
	METHOD 2 Assume commutativity, so (eg) $ap = pa$ **(i)** $\Rightarrow$ $p = ap.a \Rightarrow p = pa.a = pa^2 = pp^2 = p^3$ But p and p^3 are distinct $\Rightarrow Q$ is non-commutative	 M1 M1 B1 A1		 For setting up proof by contradiction For using **(i)** and/or given properties For obtaining and stating a contradiction For stating *Q* non-commutative, with a clear argument
			$\boxed{15}$	