Edexcel Mechanics M4 – January 2005 **Answers**

1. (a) Impulse on A is in the direction of the line of centres.

Impulse on A $= \Delta(mv) = m(-2\mathbf{i}+5\mathbf{j}) - m(\mathbf{i}+2\mathbf{j}) = m(-3\mathbf{i}+3\mathbf{j})$.

Therefore direction of line of centres is $(-\mathbf{i}+\mathbf{j})$. A unit vector in this direction is $\dfrac{(-\mathbf{i}+\mathbf{j})}{\sqrt{2}}$.

(b) Let velocity of b after collision be $v_1\mathbf{i}+v_2\mathbf{j}$

Momentum conserved: $m(\mathbf{i}+2\mathbf{j}) + 5m(-\mathbf{i}+3\mathbf{j}) = m(-2\mathbf{i}+5\mathbf{j}) + 5m(v_1\mathbf{i}+v_2\mathbf{j})$

Cancel m and equate coefficients: $\mathbf{i}$: $-4 = -2+5v_1$ $\quad v_1 = -\frac{2}{5}$

$\mathbf{j}$: $17 = 5+5v_2$ $\quad v_2 = \frac{12}{5}$

Velocity of B after collision $= -\frac{2}{5}\mathbf{i} + \frac{12}{5}\mathbf{j}$.

2. (a) Velocity of wind relative to man $= \mathbf{V}_{WM} = \mathbf{V}_W - \mathbf{V}_M$. $\therefore v(3\mathbf{i}-4\mathbf{j}) = \mathbf{V}_W - u\mathbf{j}$

Similarly $w\mathbf{i} = \mathbf{V}_W - \frac{1}{5}u(-3\mathbf{i}+4\mathbf{j})$.

Equate the two expressions for $\mathbf{V}_M$ that these produce: $v(3\mathbf{i}-4\mathbf{j}) + u\mathbf{j} = w\mathbf{i} + \frac{1}{5}u(-3\mathbf{i}+4\mathbf{j})$

Equate coefficients: $\mathbf{i}$: $-3v = w - \frac{3}{5}u$

$\mathbf{j}$: $-4v + u = \frac{4}{5}u$ $\quad \therefore v = \frac{1}{20}u$

(b) $\mathbf{V}_W = \frac{1}{20}u(3\mathbf{i}-4\mathbf{j}) + u\mathbf{j} = \frac{1}{20}u(3\mathbf{i}+16\mathbf{j})$

3. Treat B when $t=0$ as the origin.

$\mathbf{r}_A = 12t\mathbf{i} + 4\mathbf{j}$. $\quad \mathbf{r}_A = 16t\left(\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}\right)$

$\mathbf{BA} = \mathbf{r}_A - \mathbf{r}_B = \mathbf{i}(12t - 8t\sqrt{3}) + \mathbf{j}(4-8t)$

Length of $\mathbf{AB}$ $= \sqrt{\left((12t-8t\sqrt{3})^2 + (4-8t)^2\right)} = \sqrt{\left((144-192\sqrt{3}+192)t^2 + 16 - 64t + 64t^2\right)}$

Minimum when derivative of terms inside square root = 0:

$2t(144-192\sqrt{3}+192) - 64 + 128t = 0,\ t \approx 0.47$. (Minimum because this is a +ve quadratic.)

Substitute back into length of $\mathbf{AB}$: $|\mathbf{AB}| \approx 0.90$ km.

4. Apply $F = ma$: $m\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{RU}{v} - R$

Separate the variables: $\int_{\frac{u}{4}}^{\frac{u}{2}} \frac{mv\mathrm{d}v}{R(U-v)} = \int_0^T \mathrm{d}t$.

$$\frac{m}{R}\int_{\frac{u}{4}}^{\frac{u}{2}}\left(-1+\frac{U}{(U-v)}\right)\mathrm{d}v = [t]_0^T = T$$

$$\frac{m}{R}\left[-v - U\ln|U-v|\right]_{\frac{u}{4}}^{\frac{u}{2}} = T$$

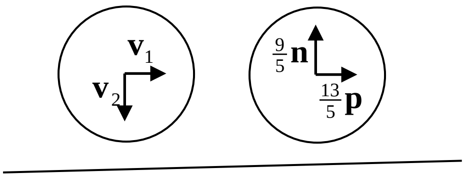
$$T = \frac{m}{R}\left(\left(-\tfrac{1}{2}U - U\ln\left(\tfrac{1}{2}U\right)\right) - \left(-\tfrac{1}{4}U - U\ln\left(\tfrac{3}{4}U\right)\right)\right)$$

$$T = \frac{mU}{R}\left(-\tfrac{1}{4} + \ln\left(\tfrac{3}{2}\right)\right)$$

5. (Note: error in question: $\mathbf{p} = +\tfrac{4}{5}\mathbf{i} + \tfrac{3}{5}\mathbf{j}$.)

(a) $\tfrac{9}{5}\mathbf{n} + \tfrac{13}{5}\mathbf{p} = \tfrac{9}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right) + \tfrac{13}{5}\left(\tfrac{4}{5}\mathbf{i} + \tfrac{3}{5}\mathbf{j}\right) = \tfrac{25}{25}\mathbf{i} + \tfrac{75}{25}\mathbf{j} = \mathbf{i} + 3\mathbf{j}$

(b) Before After



No impulse parallel to the wall so velocity parallel to wall unchanged: $\mathbf{v}_1 = \tfrac{13}{5}\mathbf{p}$

Newton's law of Restitution perpendicular to the wall: $e\mathbf{v}_2 = -\tfrac{9}{5}\mathbf{n}$

Put in values: $\tfrac{9}{16}\mathbf{v}_2 = -\tfrac{9}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right)$, $\mathbf{v}_2 = -\tfrac{16}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right) = \tfrac{39}{25}\mathbf{i} - \tfrac{64}{25}\mathbf{j}$

$\mathbf{v}_1 + \mathbf{v}_2 = \tfrac{13}{5}\left(\tfrac{4}{5}\mathbf{i} + \tfrac{3}{5}\mathbf{j}\right) - \tfrac{16}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right) = 4\mathbf{i} - \mathbf{i}$

(c) Change in KE $= \tfrac{1}{2}\times\tfrac{1}{2}\times\left(4^2 + 1^2\right) - \tfrac{1}{2}\times\tfrac{1}{2}\times\left(3^2 + 1^2\right) = 1.75$ J

6. (a) Take O as zero p.e.

Mechanical potential energy (mgh) $= -mga\cos 2\theta$

Elastic potential energy $\left(\frac{\lambda x^2}{2l}\right)$ $= \frac{1}{2} \times \frac{4mg}{\frac{5}{4}a}\left(2a\cos 2\theta - \frac{5}{4}a\right)^2$

Total p.e. $= -mga\left(2\cos^2\theta - 1\right) + \frac{8mg}{5a}\left(\frac{8a\cos\theta - 5a}{4}\right)^2$

$$= -2mga\cos^2\theta + mga + \frac{mga}{10}(8\cos\theta - 5)^2$$

$$= \frac{mga}{10}(8\cos\theta - 5)^2 - 2mga\cos^2\theta + c$$

(change of constant with referral of p.e. to any other zero position.)

(b) Equilibrium when p.e. is max/min so $\frac{\mathrm{d}E}{\mathrm{d}\theta} = 0$

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = \frac{mga}{10} \times 16 \times (8\cos\theta - 5)(-\sin\theta) + 4mga\cos\theta\sin\theta = 0$$

$$mga\sin\theta\left(-\tfrac{64}{5}\cos\theta + 8 + 4\cos\theta\right) = 0$$

$$mga\sin\theta\left(8 - \tfrac{44}{5}\cos\theta\right) = 0$$

$\sin\theta = 0$, $\theta = 0$

or

$\cos\theta = \frac{10}{11}$, $\theta = 24.6°$

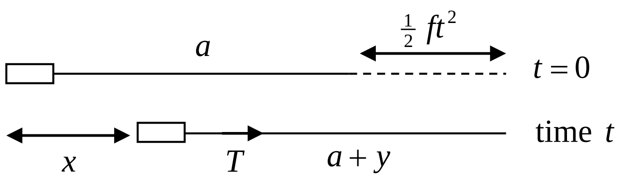
(c) $\frac{\mathrm{d}^2E}{\mathrm{d}\theta^2} = mga\sin\theta\left(\tfrac{44}{5}\sin\theta\right) + mga\cos\theta\left(8 - \tfrac{44}{5}\cos\theta\right)$

When $\theta = 0$, $\frac{\mathrm{d}^2E}{\mathrm{d}\theta^2} = mga\left(8 - \tfrac{44}{5}\right) = -\tfrac{4}{5}mga$ which is < 0 so max E so unstable.

When $\theta = 24.6°$, $\frac{\mathrm{d}^2E}{\mathrm{d}\theta^2} = mga\left(\tfrac{44}{5}\left(1 - \left(\tfrac{10}{11}\right)^2\right) + \tfrac{10}{11}\left(8 - \tfrac{44}{5} \times \tfrac{10}{11}\right)\right) = \tfrac{84}{55}mga$

which is > 0 so min E so stable.

7. (a)



At time t:

- the particle has moved x,
- the string is length $(a+y)$,
- the end of the string has moved $\frac{1}{2}ft^2$. $\therefore a+\frac{1}{2}ft^2 = x+a+y,\ \therefore x+y=\frac{1}{2}ft^2$.

(b) $F=ma$ to particle: $T=m\ddot{x}$

$$\frac{man^2}{a}y = m\ddot{x}$$

$$n^2\left(\tfrac{1}{2}ft^2 - x\right) = \ddot{x}$$

$$\ddot{x}+n^2x=\tfrac{1}{2}n^2ft^2$$

(c) $x=0,\ t=0\ \therefore 0 = A-\dfrac{f}{n^2},\ \ A=\dfrac{f}{n^2}$

Differentiating the solution: $\dot{x} = -nA\sin nt + nB\cos nt + ft$ (or get $\ddot{x}$ from original d.e.)

$\dot{x}=0,\ t=0\ \therefore 0=nB,\ \ B=0$

(d) Differentiating again: $\ddot{x} = -n^2A\cos nt - n^2B\sin nt + f = -f\cos nt + f$

$T = m\ddot{x} = mf(1-\cos nt)$.

This takes max value of $2mf$ (when $\cos nt = -1$).