OCR
RECOGNISING ACHIEVEMENT

GCE

Mathematics

Advanced GCE

Unit **4726:** Further Pure Mathematics 2

Mark Scheme for June 2011

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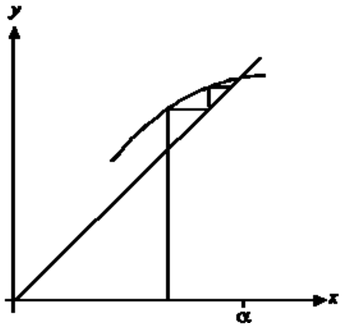
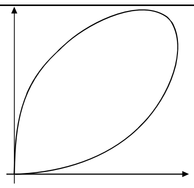
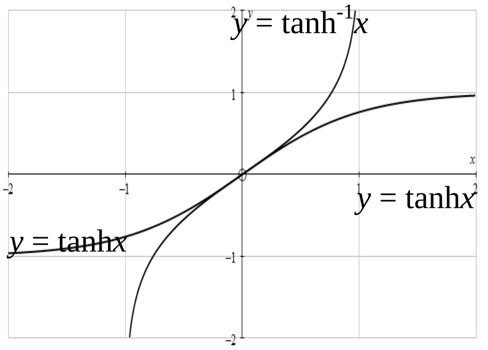
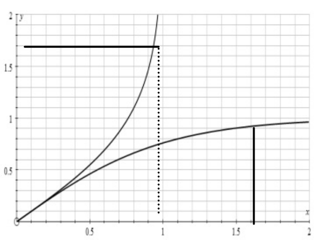
1	$\dfrac{2x+3}{(x+3)(x^2+9)} \equiv \dfrac{A}{x+3}+\dfrac{Bx+C}{x^2+9}$ $A=-\dfrac{1}{6}$ $2x+3 \equiv A(x^2+9)+(Bx+C)(x+3)$ $B=\dfrac{1}{6}, \quad C=\dfrac{3}{2}$ $\Rightarrow \dfrac{-1}{6(x+3)}+\dfrac{x+9}{6(x^2+9)}$	**B1** **B1** **M1** **A1** **A1** **5**	For correct form seen anywhere with letters or values For correct *A* (cover up or otherwise) For equating coefficients at least once.(or substituting values) into correct identity. For correct *B* and *C* For correct final statement cao, oe
2(i)	Asymptote $x = 2$ $y = x-4-\dfrac{13}{x-2}$ $\Rightarrow$ asymptote $y = x-4$	**B1** **M1** **A1** **3**	For correct equation For dividing out (remainder not required) For correct equation of asymptote (ignore any extras)
(ii)	METHOD 1 $x^2-(y+6)x+(2y-5)=0$ $b^2-4ac\,(\geq 0) \Rightarrow (y+6)^2-4(2y-5)(\geq 0)$ $\Rightarrow y^2+4y+56(\geq 0)$ $\Rightarrow (y+2)^2+52 \geq 0$: this is true $\forall y$ So y takes all values	 **M1** **M1** **A1** **A1**	**N.B. answer given** For forming quadratic in *x* For considering discriminant For correct simplified expression in *y* **soi** For completing square (or equivalent) and correct conclusion **www**
	METHOD 2 Obtain $\dfrac{dy}{dx}=\dfrac{x^2-4x+17}{(x-2)^2}$ OR $1+\dfrac{13}{(x-2)^2}$ $\Rightarrow \dfrac{dy}{dx} \geq 1 \ \forall x$, so y takes all values.	**M1** **A1** **M1** **A1** **4**	For finding $\dfrac{dy}{dx}$ either by direct differentiation or dividing out first For correct expression **oe.** For drawing a conclusion For correct conclusion **www**
	Alternate scheme: Sketching graph Graph correct approaching asymptotes from both side Graph completely correct Explanation about no turning values Correct conclusion	 **B1** **B1** **B1** **B1**	A graph with no explanation can only score 2

3(i)	$x_1 = 3.1 \Rightarrow x_2 = 3.13140$, $x_3 = 3.14148$	**B1** **B1** **2**	For correct x_2 For correct x_3
(ii)	$F'(\alpha) \approx \frac{e_3}{e_2} = \frac{0.00471}{0.01479} = 0.318\ (0.31846)$ $F'(\alpha) = \frac{1}{\alpha} = 0.3178\ (0.31784)$	**M1** **A1** **B1** **3**	For dividing e_3 by e_2 For estimate of $F'(\alpha)$ For true $F'(\alpha)$ obtained from $\frac{\mathrm{d}}{\mathrm{d}x}(2+\ln x)$ **TMDP anywhere in (i) (ii) deduct 1 once (but answers must round to given values or A0)**
(iii)	 Staircase	**B1** **B1** **B1** **3**	For $y = x$ and $y = F(x)$ drawn, crossing as shown For lines drawn to illustrate iteration (Min 2 horizontal and 2 vertical seen) For stating “staircase”

4(i)	$x = r\cos\theta,\ y = r\sin\theta$ $\Rightarrow r = \dfrac{a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$ for $0 \le \theta \le \frac{1}{2}\pi$	**M1** **A1** **A1** **3**	For substituting for x and y For correct equation **oe** (Must be r = …..) For correct limits for θ (Condone <)
(ii)	$f\left(\frac{1}{2}\pi - \theta\right) = \dfrac{a\cos\left(\frac{1}{2}\pi - \theta\right)\sin\left(\frac{1}{2}\pi - \theta\right)}{\cos^3\left(\frac{1}{2}\pi - \theta\right) + \sin^3\left(\frac{1}{2}\pi - \theta\right)}$ $= \dfrac{a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$ $f(\theta) = f\left(\frac{1}{2}\pi - \theta\right) \Rightarrow \alpha = \frac{1}{4}\pi$	**M1** **A1** **A1** **3**	**N.B. answer given** For replacing θ by $\left(\frac{1}{2}\pi - \theta\right)$ in their $f(\theta)$ For correct simplified form. (Must be convincing) For correct reason for $\alpha = \frac{1}{4}\pi$
(iii)	$r = \dfrac{a.\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3} = \frac{1}{2}\sqrt{2}\,a$	**B1** **1**	For correct value of r.**oe**
(iv)		**B1** **B1** **2**	Closed curve in 1st quadrant only, symmetrical about $\theta = \frac{1}{4}\pi$ Diagram showing $\theta = 0, \frac{1}{2}\pi$ tangential at O

5(i)	$x=\sin y \Rightarrow \dfrac{dx}{dy}=\cos y$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-\sin^2 y}}=\dfrac{1}{\sqrt{1-x^2}}$ $+\sqrt{\ }$ taken since $\sin^{-1}x$ has positive gradient	**M1** **A1** **B1** **3**	For implicit diffn to $\dfrac{dy}{dx}=\pm\dfrac{1}{\cos y}$ **oe** For using $\sin^2 y+\cos^2 y=1$ to obtain **N.B. Answer given** For justifying + sign
(ii)	$f(0)=0,\ f'(0)=1$ $f''(x)=\dfrac{x}{\left(1-x^2\right)^{\frac{3}{2}}}$ $f'''(x)=\dfrac{\left(1-x^2\right)^{\frac{3}{2}}+3x^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$ $\Rightarrow f''(0)=0,\ \ f'''(0)=1$ $\Rightarrow \sin^{-1}x=x+\frac{1}{6}x^3$	**B1** **M1** **M1** **A1** **A1** **5**	For correct values Use of chain rule to differentiate $f'(x)$ Use of quotient or product rule to differentiate $f''(0)$. For correct values **www, soi** For correct series (allow 3!) **www**
	Alternative Method: $f(0)=0,\ f'(0)=1$ $f'(x)=\dfrac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-\frac{1}{2}}=1+\dfrac{1}{2}x^2+\dfrac{3}{8}x^4+\ldots$ $f''(x)=x+\dfrac{3}{2}x^3+\ldots$ $f'''(x)=1+\dfrac{9}{2}x^2+\ldots$ $\Rightarrow f'(0)=1, f''(0)=0, f'''(0)=1$ $\Rightarrow \sin^{-1}x=x+\frac{1}{6}x^3$	**B1** **M1** **M1** **A1** **A1**	For correct values Correct use of binomial Differentiate twice Correct values Correct series
(iii)	$\left(\sin^{-1}x\right)\ln(1+x)$ $=\left(x+\frac{1}{6}x^3\right)\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3\right)$ $=x^2-\frac{1}{2}x^3+\frac{1}{2}x^4$	**B1ft** **M1** **A1** **A1** **4**	For terms in both series to at least x^3 f.t. from their **(ii)** multiplied together For multiplying terms to at least x^3 For correct series up to x^3 **www** For correct term in x^4 **www**

6(i)	$I_n = \int_0^1 x^n (1-x)^{\frac{3}{2}}\,dx$	**M1**	For integrating by parts (correct way round)
	$= \left[-\frac{2}{5} x^n (1-x)^{\frac{5}{2}} \right]_0^1 + \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{\frac{5}{2}}\,dx$	**A1**	For correct first stage
	$\Rightarrow I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{\frac{5}{2}}\,dx$	**A1**	
	$\Rightarrow I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)(1-x)^{\frac{3}{2}}\,dx$	**M1**	For splitting $(1-x)^{5/2}$ suitably
	$\Rightarrow I_n = \frac{2}{5} n I_{n-1} - \frac{2}{5} n I_n$	**A1**	For obtaining correct relation between I_n and I_{n-1}
	$\Rightarrow I_n = \frac{2n}{2n+5} I_{n-1}$	**A1** **6**	For correct result (**N.B. answer given**)
(ii)	$I_0 = \left[-\frac{2}{5} (1-x)^{\frac{5}{2}} \right]_0^1 = \frac{2}{5}$	**M1**	For evaluating I_0 [*OR* I_1 by parts]
		M1	For using recurrence relation 3 [*OR* 2] times (may be combined together)
	$I_3 = \frac{6}{11} I_2 = \frac{6}{11} \times \frac{4}{9} I_1 = \frac{6}{11} \times \frac{4}{9} \times \frac{2}{7} I_0$	**A1**	For 3 [*OR* 2] correct fractions
	$I_3 = \frac{32}{1155}$	**A1** **4**	For correct exact result

7(i)	 $y = \tanh^{-1}x$	**B1** **B1** **B1** **B1** **4**	Both curves of the correct shape (ignore overlaps) and labelled gradient = 1 at $x = 0$ stated For asymptotes $y = \pm 1$ and $x = \pm 1$ (or on sketch) Sketch all correct
(ii)	$\int_0^k \tanh x \,\mathrm{d}x = [\ln(\cosh x)]_0^k = \ln(\cosh k)$	**M1** **A1** **2**	For substituting limits into $\ln\cosh x$ For correct answer
(iii)	 Areas shown are equal: $x = \tanh k$ $\Rightarrow y = k$ $\Rightarrow \int_0^{\tanh k} \tanh^{-1} x \,\mathrm{d}x$ = rectangle ($k \times \tanh k$)– **(ii)** $= k \tanh k - \ln(\cosh k)$	**M1** **A1** **M1** **A1** **4**	For consideration of areas For sufficient justification For subtraction from rectangle For correct answer **N.B. answer given** **Alternative:** Otherwise by parts, as $1 \times \tanh^{-1} x$ *OR* $1 \times \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

PTO for alternative schemes

7(iii)	Alternative method 1 By parts: $I=\int_0^{\tanh k}\tanh^{-1}x\ \mathrm{d}x$ $u=\tanh^{-1}x \quad \mathrm{d}v=\mathrm{d}x$ $\mathrm{d}u=\frac{1}{1-x^2}\mathrm{d}x \quad v=x$ $\Rightarrow I=\left[x\tanh^{-1}x\right]_0^{\tanh k}-\int_0^{\tanh k}\frac{x}{1-x^2}\ \mathrm{d}x$ $=k\tanh k+\frac{1}{2}\left[\ln(1-x^2)\right]_0^{\tanh k}$ $=k\tanh k+\frac{1}{2}\ln(1-\tanh^2 k)$ $=k\tanh k+\frac{1}{2}\ln(\mathrm{sech}^2 k)$ $=k\tanh k+\ln(\mathrm{sech}k)$	**M1** **A1** **M1** **A1**	For integrating by parts (correct way round) For getting this far Dealing with the resulting integral
	Alternative method 2 By substitution Let $y=\tanh^{-1}x\Rightarrow x=\tanh y$ $\Rightarrow \mathrm{d}x=\mathrm{sech}^2 y\ \mathrm{d}y$ When $x=0,\ y=0$ When $x=\tanh k,\ y=k$ $\Rightarrow I=\int_0^{\tanh k}\tanh^{-1}x\ \mathrm{d}x=\int_0^k y\,\mathrm{sech}^2 y\ \mathrm{d}y$ $u=y \quad \mathrm{d}v=\mathrm{sech}^2 y\ \mathrm{d}y$ $\mathrm{d}u=\mathrm{d}y \quad v=\tanh y$ $\Rightarrow I=\left[y\tanh y\right]_0^k-\int_0^k\tanh y\ \mathrm{d}y$ $=k\tanh k-\ln\cosh k$	**M1** **A1** **M1** **A1**	For substitution to obtain equivalent integral Correct so far For integration by parts (correct way round) Final answer

8(i)	$x = \cosh^2 u \Rightarrow \mathrm{d}u = 2\cosh u \sinh u \,\mathrm{d}u$	**B1**	For correct result
	$\int \sqrt{\frac{x}{x-1}}\,\mathrm{d}x = \int \frac{\cosh u}{\sinh u} 2\cosh u \sinh u \,\mathrm{d}u$	**M1**	For substituting throughout for x
	$= \int 2\cosh^2 u \,\mathrm{d}u$	**A1**	For correct simplified u integral
	$= \int (\cosh 2u + 1)\,\mathrm{d}u = \sinh u \cosh u + u$	**M1**	For attempt to integrate $\cosh^2 u$
		A1	For correct integration
	$= x^{\frac{1}{2}}(x-1)^{\frac{1}{2}} + \ln\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right) (+c)$	**M1**	For substituting for u
		A1 **7**	For correct result **oe** as $\mathrm{f}(x) + \ln(\mathrm{g}(x))$
(ii)	$2\sqrt{3} + \ln(2+\sqrt{3})$	**B1** **1**	
(iii)	$V = (\pi)\int_1^4 \frac{x}{x-1}\,\mathrm{d}x = (\pi)\left[x + \ln(x-1)\right]_1^4$	**M1**	For attempt to find $\int \frac{x}{x-1}\,\mathrm{d}x$
		A1	For correct integration (ignore π)
	$V \to \infty$	**B1** **3**	For statement that volume is infinite (independent of M mark)

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